Application of Python 3.9 framework to investigate the dynamics of vortex-solitons

Aplicación de Python 3.9 en un sistema de investigación para la dinámica de los vórtices-solitones

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PALABRAS CLAVE:

Fibra, solitones, ecuación de Schrödinger no lineal, dinámica

RESUMEN

En nuestro estudio, la estabilidad de los solitones de la ecuación de Schrodinger no lineal se investiga mediante el cálculo de la dinámica de los anillos de vórtice. Para hacer eso usamos el paquete PY-PDE moderno en el marco de Python 3.9. El propósito principal de este paquete es simular las ecuaciones diferenciales parciales (PDE) en geometría simple. La evolución temporal de la PDE se determina utilizando el método de línea mediante muestreo explícito del espacio, utilizando cuadrículas fijas con visualización temporal simultánea de la dinámica. Dicho sistema admite el uso de programación orientada a objetos y el método compilado con números y variables para acelerar los cálculos. Con el uso del paquete PY-PDE investigamos numéricamente el fenómeno dinámico en un condensado de Bose-Einstein (BEC) colocado en un potencial externo. El mismo modelo también se puede aplicar para estudiar los solitones electromagnéticos espaciales en fibras. Se consideran tanto los casos unidimensionales como los bidimensionales.

KEYWORDS:

Fiber, solitons, nonlinear Schrodinger equation, dynamics

ABSTRACT

In our study the stability of solitons of the nonlinear Schrodinger equation is investigated via the computation the dynamics of vortex rings. To do that we use the modern PY-PDE package in the framework of Python 3.9. The main purpose of this package is to simulate the partial differential equations (PDE) in simple geometry. Temporal evolution of the PDE is determined using the line method by explicitly sampling space, using fixed grids with simultaneous temporal visualization of dynamics. Such a system supports the use of object-oriented programming and the numba-compiled method to speed up computations. With the use of PY-PDE package we numerically investigate the dynamic phenomenon in a Bose-Einstein condensate (BEC) placed in an external potential. The same model can also be applied to study the spatial electromagnetic solitons in fibers. Both the one-dimensional and two-dimensional cases are considered.

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1. INTRODUCTION

Nowadays, solitons have an important role in physics and mathematics [1]. A **soliton** or **solitary wave** is a self-reinforcing wave packet that maintains its shape while it propagates at a constant velocity [1]. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. Solitons (and vortices-solitons) are the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems ([2], [3]) Unlike 1D solitons, which are typically stable, 2D and 3D ones are vulnerable to instabilities [4] induced by the occurrence of the critical and supercritical collapse, respectively, in the 2D and 3D models with the cubic self-focusing nonlinearity. Vortex solitons are subject to a still stronger splitting instability. For this reason, a central problem is looking for physical settings in which 2D and 3D solitons may be stabilized [4]. One of the most important studies in solitons has been the study of their different dynamics (see [2]-[6]) and references therein, especially those that refer to 2D setup. In the last 15 years a challenging subject in the study of dynamic patterns in Bose-Einstein condensates (BECs) is the investigation of matter-wave solitons in multidimensional settings [3]. Our study addresses of well-established topics, the stabilization of vortex solitons by means of competing nonlinearities.

As known from the previous studies [4], [6]-[17] and references therein, the dependence between the chemical potential and the norm N (which is proportional to the number of atoms in BEC or total power of the optical beam) for 2D solitons supported by lattice potentials, features two branches, stable and unstable ones, respectively, according to the Vakhitov-Kolokolov (VK) criterion [2].

The stability of the solitons is investigated via the computation of instability of the vortex rings, which may be subject to the azimuthal instability, like in the single-component model.

To solve partial differential equations (PDEs), we used the PY-PDE package that provides differential operators to express the spatial derivatives [5]. These operators are implemented using the finite difference method to support various boundary conditions. The time evolution of the PDE is then calculated using the method of lines by explicitly discretizing space using the grid classes. This reduces the PDEs to a set of ordinary differential equations, which can be solved using standard methods. The finite differences scheme used by PY-PDE is currently restricted to orthogonal coordinate systems with uniform discretization. Because of the orthogonality, each axis of the grid can be discretized independently. For simplicity, we only consider uniform grids, where the support points are spaced equidistantly along a given axis, i.e., the discretization of x is constant. If a given axis covers values in a range [x_{min}; x_{max}], a discretization with N support points can then be thought of as covering the axis with N equal-sized boxes.

$$x_i = x_{min} + \left(i + \frac{1}{2}\right)\Delta x \qquad (1)$$

Differential operators are implemented using the usual second-order central difference. This requires the introducing of virtual support points at x-1 and xN, which can be determined from the boundary conditions at $x = x_{min}$ and $x = x_{max}$, respectively. With the use of PY-PDE package we numerically investigate the dynamic excitations in a Bose-Einstein condensate (BEC) placed in an external potential Different type of solitons in Bose-Einstein condensates (BECs) have been intensively studied both theoretically and experimentally ([6]-[15]). Recently, solitons supported by weak attractive interactions between atoms were created in the condensate of 7Li trapped in strongly elongated traps. Also, a possibility to design 1D and 2D quantum systems by "freezing out" one or two degrees of freedom by adding a 1D or 2D optical lattice to the magnetic trap has been demonstrated experimentally.

2. BASIC EQUATIONS

In order to trap a Bose-Einstein condensate in a (quasi)-periodic potential, it is sufficient to exploit the interference pattern created by two or more overlapping laser beams and the light force exerted on the condensate atoms. We mainly focus on the physical situation when the number of atoms is sufficiently large, so that atomic number fluctuations are negligible and the mean-field approximation can be applied. In this approach an anisotropic BEC cloud, loaded into a two-dimensional optical lattice potential V (r), is described by the Gross-Pitaevskii (GP) equation (temperature T = 0)

$$i\frac{\partial \Phi}{\partial \tau}=\left[-\nabla_{\perp}^{2}+v(r)+G|\Phi|^{2}\right]\Phi \qquad (2)$$

where $r=(x,y), \nabla_{\perp}=(\partial_x,\partial_y), \Phi$ is the condensate wave function (with one or two components) and G = ±1 for repulsive and attractive interaction respectively. This equation is obtained by assuming a tight confinement in the direction perpendicular to the lattice ("pancake" trapping geometry) and the standard dimensionality reduction procedure (see e.g. Ref.[2]). According to Eq. (2), a particle with the (condensate) wave function Φ(r; t) evolves in the external potential V (r) induced by OL plus the mean-field potential created by the remaining particle.

3. NUMERICAL RESULTS

First, we consider the simplest 1D case when the system has only dependences on coordinate x and time t and does not depends on y coordinate. Dependently on the sign of G in Eq. (2) the system has different dynamics. Fig. 1 and Fig. 2 shows the cases G<0 (attractive interaction) and G>0 in Eq. (1) respectively for the potential-free case V=0. We observe that the initial localized pulse will propagates as a single soliton along x (at G<0) or is quickly dissociated in many small fragments. Fig. 3-6 show the soliton dynamics for cases for two-well potential $V(x) = V0*(a1(x-x_0)^2 + a2 (x-x_0)^4)$. We observe that the initial localized pulse will propagate as a single soliton along x (G<0) or is quickly dissociated in many small fragments (G>0).

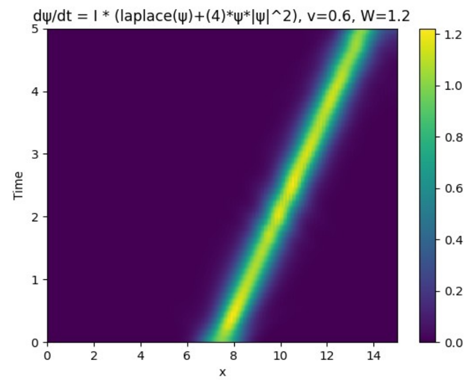

Figure 1. (Color on line.) 1D case. Attractive interaction, G<0 in Eq.(1). In this case we observe the propagation of a stable single soliton along x.

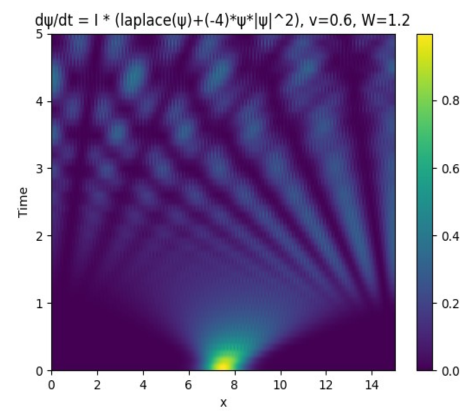

Figure 2. (Color on line) 1D case. Repulsive interaction, G>0 in Eq.(1). In this case the initial soliton is quickly dissociated into many fragments.

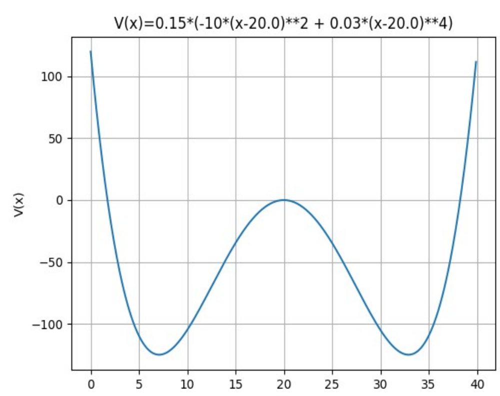

Figure 3. (Color on line) 1D case. Two-Well potential V(x) is indicated in the top of figure.

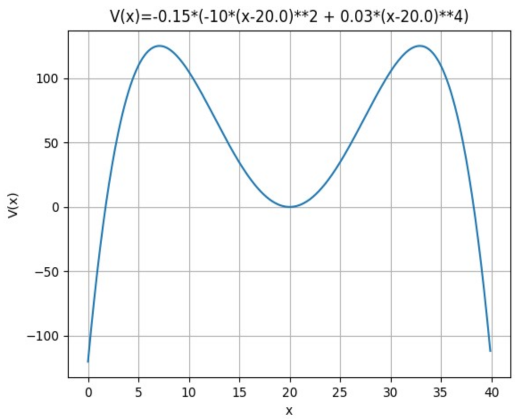

Figure 4. (Color on line.) 1D case. The inverse two-Well potent V(x) is indicated in the top of figure.

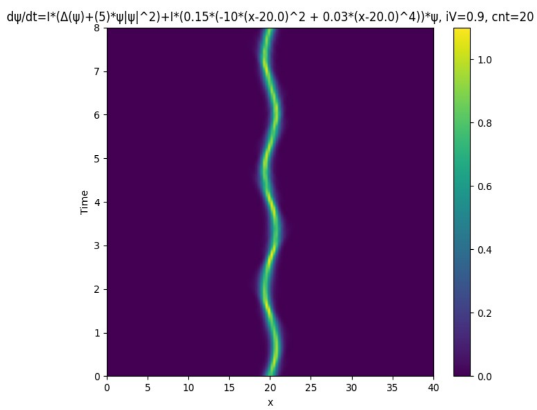

Figure 5. Propagation of the soliton in 1D case at the attractive interaction in Two-Well potential V(x) indicated in the top of figure.

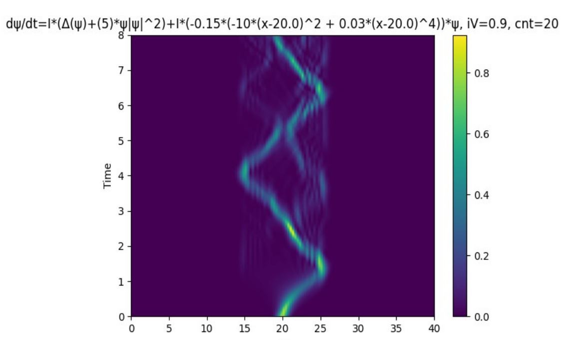

Figure 6. Propagation of the soliton in 1D case in Two-Well potential V(x) indicated in the top of figure.

All the figures shown in this paper were generated with the use of PDE-PY package for Python 3.9.

The Python code for Figs 1-6 with the use of the PDE-PY package [4] for Python 3.9:

```
from math import sqrt, cosh
import numpy as np
from pde import PDE, CartesianGrid, MemoryStorage, ScalarField,
plot_kymograph
import matplotlib.pyplot as plt

grid = CartesianGrid([[0, 2*20]], 128, periodic=False)  # generate grid

# create a (normalized) wave packet with a certain form as an initial
condition
center = grid.axes_bounds[0][1]/2
iVel=0.9    # can be changed
g = 5       # can be variated: g=5>0 is a soliton state, g=-5<0 is a
decaying state
ampPot_V0 = 0.15      # V_0 is amplitude of potential V_0*V(x)
sInitState = "exp(I * "+str(iVel)+" * x) * exp(-(x - " + str(center) +
")**2)"
print("\n\nInit state: "+ sInitState)
initial_state = ScalarField.from_expression(grid, sInitState)
initial_state /=
sqrt(initial_state.to_scalar("norm_squared").integral.real)

V_x = f""+str(ampPot_V0)+"*(-10*(x-"+str(center)+")**2 + 0.03*(x-
"+str(center)+")**4)"
#----------plot potential----
def plot2D(*args):
    plt.grid(1)
    xAxis = np.arange(args[1],args[2],args[3])
    def xfunction(x,input):
        return eval(input)
    plt.title(args[4])
    plt.plot(xAxis,xfunction(xAxis,args[0]))
    plt.xlabel("x"); plt.ylabel("V(x,t)")
    plt.show(block=False)

print("Plot potential: V(x)=" + V_x)  # V_0 * V(x)
plot2D(V_x, grid.axes_bounds[0][0], grid.axes_bounds[0][1], 0.1, "V(x)="
+ V_x)
sFun0 = f"I*(laplace(ψ)+("+str(g)+")*conjugate(ψ)*ψ**2)" # define the pde
sFun = f"{sFun0}+I*({V_x})*ψ"   # add potential V(x)
eq = PDE({"ψ": sFun})           # define the pde
# solve the pde and store intermediate data
storage = MemoryStorage()
eq.solve(initial_state, t_range=4*2, dt=1e-5,
tracker=[storage.tracker(0.02)])
# visualize the results as a space-time plot
sFun = sFun.replace("conjugate(ψ)*ψ**2","ψ|ψ|^2") # simplify the math in
title
sFun = sFun.replace("laplace","Δ")
sFun = sFun.replace("**","^")
sTitle = "dψ/dt=" + sFun + ", iV=%g" % (iVel)+ ", cnt=%g" % (center)
plot_kymograph(storage, scalar="norm_squared", title=sTitle)
print("Done.\n\n")
```

Figure. The Python code used for calculations of figures 1-6.

4. TWO DIMENSIONAL (2D) CASE

In this Section we study nontrivial two-dimension (2D) soliton dynamics in a system with qubic-quintic (CQ) nonlinearity with PT symmetry that is described by system of Eq. (3). We mainly are interesting to study the formation and dynamics of the 2D vortex solitons stabilized by means of competing CQ nonlinearities. To solve partial differential equations (PDE), that is a system of complex nonlinear Schrödinger equations we apply the PY-PDE package that provides differential operators to express spatial derivatives [5]. These operators are implemented using the finite difference method to support various boundary conditions. The finite difference scheme used by PY-PDE is currently restricted to orthogonal coordinate systems with uniform discretization. Due to the orthogonality, each axis of the grid can be discretized independently. To simplify this, we only consider uniform grids, where the support points are equally spaced.

In our simulations, we found that the package PDE-PY was able to successfully numerically investigate the advanced dynamics of the vortex solitons too. It is very important that the PDE-PY package not only calculates a numerical solution for two-dimensional complex coupled vortices, but also allows one to see their temporal nonlinear dynamics. We mention that in the general form, the model, which describes the spatio-temporal propagation of light in the dual-core planar optical wave guide, is based on a system of two-dimensional (2D) nonlinear Schrodinger (NLS) equations for amplitudes of the electromagnetic field in two cores, coupled by the linear terms, which account for the tunneling of light between the cores, where *z* is the propagation distance. Fig. 7 shows the structure of the editor window of VS Code, where also are shown the shapes of the dynamical vortex-solitons in different planes.

$$i\frac{\partial \Psi}{\partial z}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+|\Psi|^2\Psi-|\Psi|^4\Psi+\lambda\Phi=i\gamma\Psi\,, \qquad (3a)$$

$$i\frac{\partial \Phi}{\partial z}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi+|\Phi|^2\Phi-|\Phi|^4\Phi+\lambda\Psi=-i\gamma\Phi \qquad (3b)$$

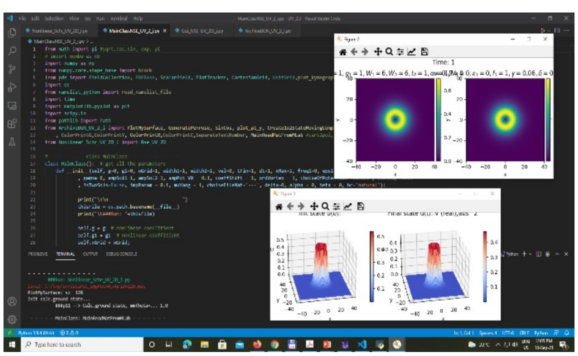

Figure 7. The structure of the editor window of VS Code, where also are shown the shapes of the dynamical vortex-solitons in the different planes.

In what follows we illustrate the result of our simulations with the use of the package PDE-PY to numerically investigate the detailed dynamics of the vortex solitons.

The nontrivial dynamics of 2D vortexes arise because of after sudden (short with respect to the scale t_0) by means of competing CQ nonlinearity.

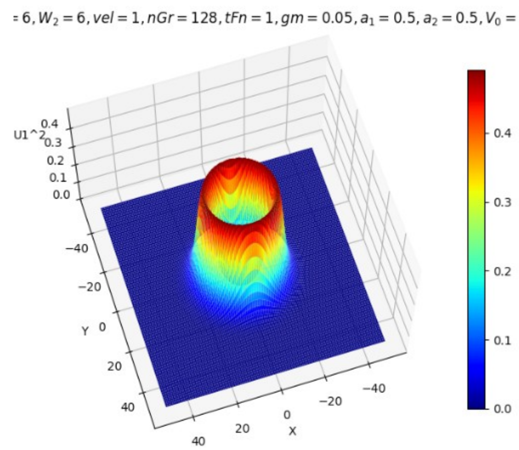

Figure 8. The initial (t=0) vortex state.

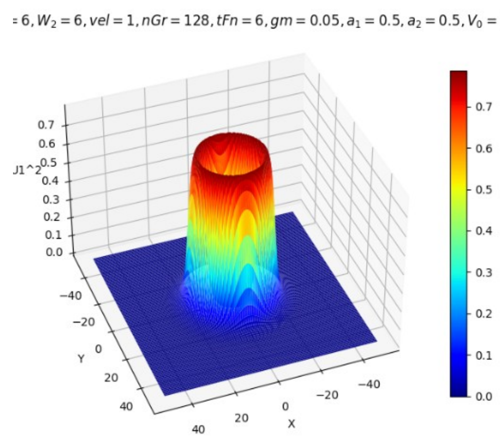

Figure 9. The same as in Fig 8 but after the simulation time t=6.

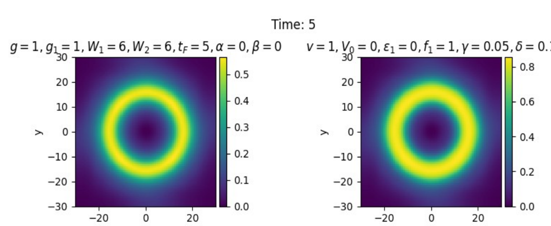

Figure 10. The same as in Fig.8 but for time t=5.

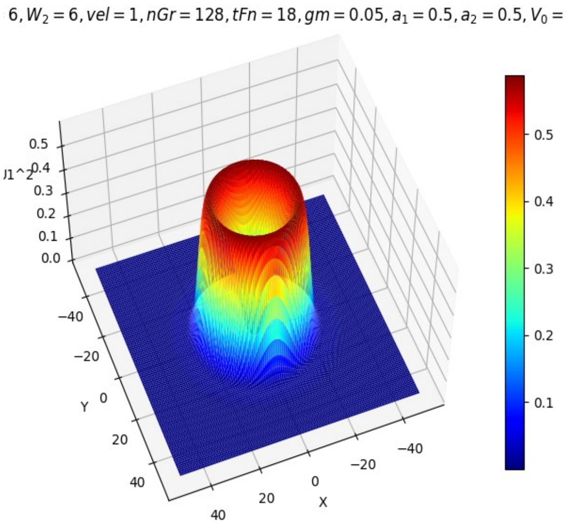

Figure 11. The same as in Fig.8 but for time t=18.

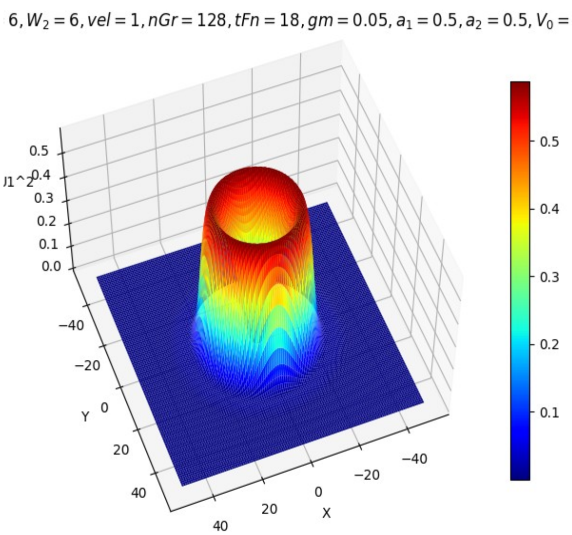

Figure.12. The same as in Fig.8 but for time t=24.

We observe that the shape of the vortexes practically does not change with the time. Thus, it is numerically confirms that the vortexes in such an advanced dynamic nonlinear system are long time-living excitations.

5. FUTURE WORK

In our future work, we plan to extend the study of the dynamics and stability of 2D and 3D solitons for more complex systems. These may be spatially inhomogeneous materials or media in which the external potential is a function of time with rapid switching of the spatial structure.

6. CONCLUSIONS

We have studied the dynamics of solitons of the nonlinear Schrodinger equation with use the modern PY-PDE package in the framework of Python 3.9. Temporal evolution of the PDE is determined using the line method by explicitly sampling space. In our investigation we used the object-oriented programming and the numba-compiled method [4] to speed up computations. With the use of PY-PDE package we numerically investigated the dynamic phenomenon in a Bose-Einstein condensate (BEC) placed in an external potential and also studied the spatial electromagnetic solitons in fibers. Both one-dimensional and two-dimensional cases are considered. It is numerically confirmed that the vortexes in such an advanced CQ nonlinear system are long-time-stable excitations.

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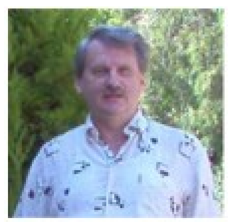

Dr. Gennadiy Burlak, CIICAp de la UAEM. El Dr. Gennadiy Burlak ha trabajado como catedrático en la Universidad Nacional de Kiev, en el Departamento de Física Teórica. Tiene los grados de doctor en: Ph. D. y D. of Sc. Desde 1998 es Profesor-Investigador Titular "C" del Centro de Investigaciones en Ingeniería y Ciencias Aplicadas (CIICAp) de la Universidad Autónoma del Estado de Morelos (UAEM). Es miembro del SNI desde 2000 y actualmente tiene el nivel III. El Dr. Burlak es autor y coautor de 14 libros y capítulos de libros y 170 artículos en revistas internacionales. Ha participado en más de 170 ponencias en Congresos Nacionales e Internacionales. Bajo de su dirección se han graduado: 16 tesis de doctorado, maestría y licenciatura. Ha impartido cursos de electromagnetismo, ecuaciones derivadas parciales y métodos numéricos en el posgrado y licenciatura del CIICAp de la UAEM. Es miembro de la Academia de Ciencias de Morelos (ACMOR) de American Physical Society. Se ha desempeñado como evaluador, árbitro del CONACyT y como referí de varias revistas internacionales como lo son: Phys.Rev.Lett., Chaos, JVSTA, MMA,PIER, entre otros. Sus temas principales de investigación son:

Micro-esféricas multicapas, Optimización de radiación óptica en nanoestructuras, Dinámica no-lineal del Bose-Einstein condénsate, Aplicaciones de redes neuronales en física cuántica y transición de fases en sistemas sólidos.

Dra. Yessica Yazmin Calderon Segura. La Dra. Yessica Yazmin Calderon Segura, cuenta con una experiencia en optimización de algoritmos, modelos matemáticos, procesos para minimizar tiempos, sistemas de percolación, nanoestructuras y fenómenos electromagnéticos. Ha publicado artículos con coautoría en revistas internacionales y con alto factor de impacto. Así como otros conocimientos en los temas de procesamiento de imágenes, redes neuronales y sistemas. Actualmente es miembro del SNI, como candidata. Autora y coautora de 7 artículos en revistas internacionales. He participado en 22 ponencias en congresos nacionales e internacionales. Bajo de su dirección han graduado: 1 tesis de licenciatura FCAeI-UAEM. Actualmente 2 tesis de licenciatura FCAeI-UAEM en proceso, bajo su dirección.